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Translated by L. K.

UDC 531.31+517.51

ON A PROBLEM OF POINCARÉ

PMM Vol. 40, № 2, 1976, pp. 352-355

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(Received November 19, 1974)

We consider the behavior of the integral

$$I(t) = \int_0^t f(\omega_1 t, \omega_2 t) dt$$

as $t \rightarrow +\infty$. Here $f(\varphi_1, \varphi_2)$ is a continuous function of a two-dimensional torus $T^2 \{ \varphi_1, \varphi_2 \pmod{1} \}$, and the ratio of the frequencies ω_2 / ω_1 is irrational. The problem was first studied by Poincaré [1] and is often encountered in analytic studies of the dynamic systems.

It is well known [1, 2] that if

$$\int_0^1 \int_0^1 f(\varphi_1, \varphi_2) d\varphi_1 d\varphi_2 > 0 (< 0)$$

then $I(t) \rightarrow +\infty (-\infty)$ as $t \rightarrow +\infty$. A difficulty arises when the mean spatial value of the function f is zero. Poincaré used examples to show in [1] that in this case the integral $I(t)$ can tend either to $+\infty$ or to $-\infty$ (like t^α , $0 < \alpha < 1$) and, in the most interesting case, be unbounded but able to approach its initial value (equal to zero) infinitely many times and as closely as required. A question naturally arises of determining the conditions under which the integral $I(t)$ will be recurrent (Poisson stability). The first step towards this solution consists of inspecting the discrete analog of the problem, and this helps us to establish that the recurrence takes place if the function f is twice continuously differentiable.

1. We assume that a continuous function $f(x)$ is given on the circumference $S^1 \{x \pmod{1}\}$. Let α be an irrational number. We construct the sum

$$S_N(\alpha, \varphi) = \sum_{i=0}^{N-1} f(i\alpha + \varphi), \quad \varphi \in S^1$$

If

$$\int_0^1 f(x) dx > 0 (< 0)$$

then obviously the sum $S_N(\alpha, \varphi) \rightarrow +\infty (-\infty)$ as $N \rightarrow \infty$ uniformly in φ .

Theorem 1. Let

$$f \in C^2(S^1) \text{ and } \int_0^1 f(x) dx = 0$$

Then for every $\varepsilon > 0$ and N_0 there exists $N > N_0$ such that $|S_N(\alpha, \varphi)| < \varepsilon$ for all $\varphi \in S^1$.

Proof. We separate all irrational numbers into two classes; class K_1 containing the numbers α for which the inequality

$$|n\alpha - m| < 1/|n|^{3/2} \tag{1.1}$$

has infinitely many integer solutions, and class K_2 containing the remaining numbers.

Let initially $\alpha \in K_2$. We expand the function f into a convergent Fourier series

$$f(x) = \sum_{-\infty}^{\infty} f_n e^{i2\pi nx} \left(|f_n| \leq \frac{c}{|n|^2}, c > 0 \right)$$

Then

$$S_N(\alpha, \varphi) = \sum_{k=0}^{N-1} \sum_{-\infty}^{\infty} f_n e^{i2\pi n(k\alpha + \varphi)} = \sum_{-\infty}^{\infty} f_n e^{i2\pi n\varphi} \frac{e^{i2\pi nN\alpha} - 1}{e^{i2\pi n\alpha} - 1} \tag{1.2}$$

Using the results of [3], we can easily show that the trigonometric series

$$\sum_{-\infty}^{\infty} \frac{f_n}{e^{i2\pi n\alpha} - 1} e^{i2\pi nx}$$

converges and represents a Fourier series of a certain continuous function $F(x)$ ($x \in S^1$).

Taking (1.2) into account, we find that $S_N(\alpha, \varphi) = F(N\alpha + \varphi) - F(\varphi)$, which makes the conclusion of the theorem obvious.

When $\alpha \in K_1$, the numbers m and n in the inequality (1.1) can be assumed to be relatively prime. If they are not, then let d ($d > 1$) be their greatest common divisor.

We set $m = dm_1$ and $n = dn_1$. Then

$$|n_1\alpha - m_1| \leq \frac{1}{d^{3/2}|n_1|^{3/2}} < \frac{1}{|n_1|^{3/2}}$$

Clearly, such a transformation will produce an infinite number of different inequalities (1.1) with relatively prime m and n . Let us estimate the difference

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} f(k\alpha + \varphi) - \sum_{k=0}^{n-1} f\left(k \frac{m}{n} + \varphi\right) \right| \leq \\ & \sum_{k=0}^{n-1} \left| f(k\alpha + \varphi) - f\left(k \frac{m}{n} + \varphi\right) \right| \leq M_1 \sum_{k=0}^{n-1} \left| k\alpha - k \frac{m}{n} \right| \leq \\ & M_1 n \sum_{k=0}^{n-1} \left| \alpha - \frac{m}{n} \right| \leq M_1 n^2 \left| \alpha - \frac{m}{n} \right| < \frac{M_1}{\sqrt{n}} \end{aligned}$$

$$M_1 = \max_{x \in S^1} |f'(x)|$$

Consequently

$$\left| \sum_{k=0}^{n-1} f(k\alpha + \varphi) \right| < \left| \sum_{k=0}^{n-1} f\left(k \frac{m}{n} + \varphi\right) \right| + \frac{M_1}{\sqrt{n}} \quad (1.3)$$

Since the numbers m and n are relatively prime, the points on S^1 the angular coordinates of which are

$$\varphi, \varphi + \frac{m}{n}, \varphi + 2 \frac{m}{n}, \dots, \varphi + (n-1) \frac{m}{n}$$

are situated at the apexes of a regular unscripted n -tuple polygon. Since $f \in C^2(S^1)$, the well known rectangular rule of numerical integration implies that a point ξ exists on the circumference S^1 such, that

$$\int_0^1 f(x) dx = \frac{1}{n} \sum_{k=0}^{n-1} f\left(k \frac{m}{n} + \varphi\right) + \frac{f''(\xi)}{24n^2}$$

From this we have

$$\left| \sum_{k=0}^{n-1} f\left(k \frac{m}{n} + \varphi\right) \right| \leq \frac{M_2}{24n}, \quad M_2 = \max_{x \in S^1} |f''(x)|$$

Taking into account (1.3), we finally obtain

$$\left| \sum_{k=0}^{n-1} f(k\alpha + \varphi) \right| < \frac{M_1}{\sqrt{n}} + \frac{M_2}{24n}$$

Since infinitely many numbers n satisfying the above inequality exist, Theorem 1 is also proved for the remaining $\alpha \in K_1$.

2. Next we consider the problem of recurrence of the integral $I(t)$.

Theorem 2. If f is continuous on T^2 , has two continuous derivatives in φ_2 and the spatial mean of f is zero, then for any $\varepsilon > 0$ and T there exists $t > T$ such that $|I(t)| < \varepsilon$.

Proof. Consider on T^2 the circumference $S^1 = \{\varphi_1, \varphi_2 : \varphi_1 = 0\}$, and the function

$$F(x) = \frac{1}{\omega_1} \int_0^1 f\left(\varphi_1, \frac{\omega_2}{\omega_1} \varphi_1 + x\right) d\varphi_1, \quad x \in S^1$$

on this circumference. We obviously have

$$I(na) = \sum_{k=0}^{n-1} F\left(k \frac{\omega_2}{\omega_1}\right) = S_n\left(\frac{\omega_2}{\omega_1}, 0\right), \quad a = \frac{1}{\omega_1} > 0$$

It can easily be shown that the function $F(x)$ satisfies the conditions of Theorem 1.

The conclusion of Theorem 2 now follows from Theorem 1.

Note. The Poincaré's example [1] in which $I(t) \rightarrow +\infty$ (or $-\infty$) does not contradict Theorem 2, since we can easily show that the function f of this example has no second derivatives in φ_1 and φ_2 .

3. We consider, as an example, the oscillations of an elastic string of length d ; let a be the rate of propagation of the perturbations. We assume that the string is stationary at the initial instant of time ($t = 0$), its left end is clamped rigidly and its right end begins to execute T -periodic oscillations in accordance with a law $f(t)$ ($f(0) = 0$). The problem of determining the forced oscillations of the string at $t \geq 0$ represents a mixed problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial x^2} \quad \{t, x: 0 \leq t < \infty, 0 \leq x \leq d\}$$

We shall denote the solution of this problem by $u(t, x)$. It can easily be shown that if the ratio $d/(aT)$ is rational, then points $x = \xi$ exist on the string such that $u(t, \xi) \rightarrow \infty$ as $t \rightarrow \infty$ (parametric resonance).

Theorem 3. Assume that $d/(aT)$ is irrational and function f belongs to class C^2 . Then for any $\varepsilon > 0$ and τ there exists $t > \tau$ such that $|u(t, x)| < \varepsilon$ for all $x \in [0, d]$.

Proof. Using the fundamental property of the characteristic parallelogram we obtain

$$u\left(2\frac{d}{a}n, x\right) = \sum_{i=1}^n f(t_{2i}) - \sum_{i=1}^n f(t_{2i-1}) \quad (3.1)$$

$$t_{2i} = t_2 + 2(i-1)\frac{d}{a}, \quad t_{2i-1} = t_1 + 2(i-1)\frac{d}{a},$$

$$t_1 = \frac{d-x}{a}, \quad t_2 = \frac{d+x}{a}$$

Let us write the function $f(t)$ in the form $c + g(t)$ where c denotes the mean value of the function $f(t)$. Then Eq.(3.1) can be rewritten as follows:

$$u\left(2\frac{d}{a}n, x\right) = \sum_{i=1}^n g(t_{2i}) - \sum_{i=1}^n g(t_{2i-1}) \quad (3.2)$$

Since the mean value of the periodic function $g(t)$ belonging to class C^2 is zero and the ratio $d/(aT)$ is irrational, by Theorem 1 for any $\varepsilon > 0$ and N_0 there exists $N > N_0$ such that

$$\left| \sum_{i=1}^N g(t_{2i}) \right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left| \sum_{i=1}^N g(t_{2i-1}) \right| < \frac{\varepsilon}{2}$$

uniformly in x . Taking into account (3.2), we conclude that the inequality $|u(t, x)| < \varepsilon$ will hold at the instant $t = 2dN/a$ for all $x \in [0, d]$.

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Translated by L. K.